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On Two Absolute Index-Summability Methods

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ABSTRACT: In this paper we have established a relation between the Summability methods $X-\left|\overline{N},p_n\right|_k$, $Y-\left|A\right|_k, k\geq 1$ and $Y-\left|A, f(\delta)\right|_k, k\geq 1, \delta\geq 0.$

Keywords: $\left|\overline{N},p_n\right|_k, X-\left|\overline{N},p_n\right|_k, X-\left|N,p_n\right|_k, X-\left|A\right|_k$ -summabilities.

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1. Introduction:

Let $\sum a_n$ be an infinite series and $\{s_n\}$the sequence of partial sums. Let $\{p_n\}$be a sequence of non-negative numbers with $P_n = \sum_{\nu=0}^{n} p_\nu$ for all $n \in N$.The sequence –to-sequence transformation

(1.1) $$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu , P_n \neq 0$$

defines $\left|\overline{N}, p_n\right|$-mean of the sequence $\{s_n\}$generated by the sequence of coefficients $\{p_n\}$.The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, **[4]** if

(1.2) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|t_n - t_{n-1}\right|^k < \infty ,$$

The sequence –to-sequence transformation

(1.3) $$\tau_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu} s_\nu , P_n \neq 0 ,$$

defines $\left|N, p_n\right|$-mean of the sequence $\{s_n\}$.The series $\sum a_n$ is said to be summable $\left|N, p_n\right|_k$, $k \geq 1$, if

(1.4) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\tau_n - \tau_{n-1}\right|^k < \infty .$$

The series $\sum a_n$ is said to be summable $X - \left|\overline{N}, p_n\right|_k$, $k \geq 1$, if

(1.5) $$\sum_{n=1}^{\infty} X_n^{k-1}\left|t_n - t_{n-1}\right|^k < \infty$$

where $\{X_n\}$is a sequence of positive real constants. Similarly, $\sum a_n$ is said to be summable $X - \left|N, p_n\right|_k$, $k \geq 1$, if

(1.6) $$\sum_{n=1}^{\infty} X_n^{k-1}\left|\tau_n - \tau_{n-1}\right|^k < \infty .$$

Let $A = (a_{nk})$ be a $\infty \times \infty$ matrix. The series $\sum a_n$ is said to be summable $X - \left|A\right|_k$, $k \geq 1$, if

(1.7) $$\sum_{n=1}^{\infty} X_n^{k-1}\left|T_n - T_{n-1}\right|^k < \infty ,$$

$\sum a_n$ is said to be summable $X - \left|A, \delta\right|_k$, $k \geq 1, \delta \geq 0$, if

(1.8) $$\sum_{n=1}^{\infty} X_n^{\delta k+k-1}\left|T_n - T_{n-1}\right|^k < \infty$$

and $\sum a_n$ is said to be summable $X - \left|A, f(\delta)\right|_k$, $k \geq 1, \delta \geq 0$, if

(1.9) $$\sum_{n=1}^{\infty}\left(f(\delta)\right)^{k} X_n^{k-1}\left|T_n - T_{n-1}\right|^{k} < \infty,$$

where the sequence –to-sequence transformation $\{T_n\}$ is given by

(1.10) $$T_n = \sum_{k=1}^{\infty} a_{nk} s_k .$$

Clearly $X - \left|A, f(\delta)\right|_k, k \geq 1, \delta \geq 0$ reduces to $X - \left|A, \delta\right|_k, k \geq 1, \delta \geq 0$ if $f(\delta) = X_n^{\delta}$.

2. Known Theorems:

Dealing with the index summability method Bor has established the following theorems:

Theorem-A[1]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$

i) $np_n = O(P_n)$ ii) $P_n = O(np_n)$.

If $\sum a_n$ is summable $\left|C,1\right|_k$ then it is summable $\left|\overline{N}, p_n\right|_k, k \geq 1$.

Theorem-B[2]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$

i) $np_n = O(P_n)$ ii) $P_n = O(np_n)$.

If $\sum a_n$ is summable $\left|\overline{N}, p_n\right|_k$ then it is summable $\left|C,1\right|_k, k \geq 1$.

Subsequently Bor and Thorpe established the following result.

Theorem-C[3]:

Let $\{p_n\}$ and $\{q_n\}$ be the sequence of positive real constants such that

i) $p_n Q_n = O(P_n q_n)$ ii) $P_n q_n = O(p_n Q_n)$.

then the series $\sum a_n$ is summable $\left|\overline{N}, q_n\right|_k$ whenever it is summable $\left|\overline{N}, p_n\right|_k, k \geq 1$.

Further, Tripathi established

Theorem-D[6]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

i) $q_n p_n = O(p_n q_n)$ ii) $Q_n = O(q_n X_n)$ iii) $Y_n p_n = O(p_n)$.

If $\sum a_n$ is summable $X - \left|\overline{N}, p_n\right|_k$, then it is summable $Y - \left|\overline{N}, q_n\right|_k, k \geq 1$.

Extending the above result , Misra, Misra and Routa established the following theorem replacing $Y - \left|\overline{N}, q_n\right|_k, k \geq 1$ by $Y - \left|N, q_n\right|_k, k \geq 1$, in Theorem-D.

Theorem-E[7]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

i) $q_n p_n = O(p_n q_n)$ ii) $Q_n = O(q_n X_n)$ iii) $Y_n p_n = O(p_n)$

iv) $\sum_{n=\nu+1}^{n+1} \frac{1}{Q_n} = O\left(\frac{1}{Q_\gamma}\right)$ v) $\sum_{\nu=1}^{n-1} a_\nu^{\frac{k}{k-1}} = O(1), k \neq 1.$

If $\sum a_n$ is summable $X - \left|\overline{N}, p_n\right|_k$, then it is summable $Y - \left|N, q_n\right|_k$, $k \geq 1$.

Further, generalizing the above theorem to matrix summability Padhy, panda, Misra and Misra established the following theorem:

Theorem-F[5]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

i) $a_{nn}P_n = O(p_n)$, ii) $\frac{1}{a_{nn}} = O(X_n)$, iii) $Y_n p_n = O(P_n)$,

iv) $\sum_{n=r}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} A_{nr} = O(a_{rr})$, where $A_{nk} = \sum_{\nu=k}^{n} a_{n\nu}$

and, for the infinite triangular matrix $A = (a_{nk})_{\infty x \infty}$,

v) $\sum_{r=1}^{n} A_{nr} = O(1)$.

Then $\sum a_n$ is $Y - \left|A, \delta\right|_k$, $k \geq 1, \delta \geq 0$ summable whenever $\sum a_n$ is summable $X - \left|\overline{N}, p_n\right|_k$, $k \geq 1$.

In what follows, in the present paper, further generalizing the above theorem we establish

3. Main Theorem:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

(3.1) i) $a_{nn}P_n = O(p_n)$,

(3.2) ii) $\frac{1}{a_{nn}} = O(X_n)$,

(3.3) iii) $Y_n p_n = O(P_n)$,

(3.4) iv) $\sum_{n=r}^{m+1} \left(f(\delta)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} A_{nr} = O(a_{rr})$, where $A_{nk} = \sum_{\nu=k}^{n} a_{n\nu}$

and, for the infinite triangular matrix $A = (a_{nk})_{\infty x \infty}$,

(3.5) v) $\sum_{r=1}^{n} A_{nr} = O(1)$.

Then $\sum a_n$ is $Y - \left|A, f(\delta)\right|_k$, $k \geq 1, \delta \geq 0$ summable, whenever $\sum a_n$ is summable $X - \left|\overline{N}, p_n\right|_k$, $k \geq 1$.

4. Proof of The Theorem:

If $\{t_n\}$ is the nth $\left|\overline{N}, p_n\right|$-mean of $\sum a_n$,then

$$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu$$

$$= \frac{1}{P_n}\{p_0 a_0 + p_1(a_0 + a_1) + \cdots + p_n(a_0 + a_1 + \cdots + a_n)\}$$

$$= \frac{1}{P_n}\{P_n a_0 + (P_n - p_0)a_1 + \cdots + (P_n - p_{n-1})a_n\}$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu$$

Then

$$\Delta t_n = t_n - t_{n-1}$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu - \frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1}(P_{n-1} - P_{\nu-1})a_\nu$$

$$= \frac{1}{P_n}\sum_{\nu=1}^{n}(P_n - P_{\nu-1})a_\nu - \frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1}(P_{n-1} - P_{\nu-1})a_\nu$$

$$= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

Hence,

$$\frac{P_n P_{n-1}}{p_n}\Delta t_n = \sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

and

$$\frac{P_{n-1}P_{n-1}}{p_{n-1}}\Delta t_{n-1} = \sum_{\nu=1}^{n-1} P_{\nu-1}a_\nu$$

Thus,

$$a_n = \frac{P_n}{p_n}\Delta t_n - \frac{P_{n-2}}{p_{n-1}}\Delta t_{n-1}$$

Further,if $\{T_n\}$ is the nth A-mean of $\sum a_n$, where $A = (a_{nk})_{\infty x \infty}$ triangular matrix,

$$T_n = \sum_{k=0}^{n} a_{nk} s_k$$

$$= \sum_{k=0}^{n} A_{nk} a_k \ , A_{nk} = \sum_{\nu=k}^{n} a_{n\nu}$$

Then,

$$T_n - T_{n-1} = \sum_{k=0}^{n} A_{nk} a_k - \sum_{k=0}^{n-1} A_{n-1,k} a_k$$

$$= \sum_{k=1}^{n} (A_{nk} - A_{n-1,k}) a_k$$

$$= \sum_{k=1}^{n} (A_{nk} - A_{n-1,k}) \left(\frac{P_k}{p_k} \Delta t_k - \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1} \right)$$

$$= \sum_{k=1}^{n} A_{nk} \frac{P_k}{p_k} \Delta t_k - \sum_{k=1}^{n} A_{nk} \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1} - \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_k}{p_k} \Delta t_k + \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1}$$

$$= S_1 + S_2 + S_3 + S_4 \text{ (say).}$$

Now,

$$\sum_{n=1}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} |T_n - T_{n-1}|^k \le \sum_{n=1}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} |S_1 + S_2 + S_3 + S_4|^k = \sum_{i=1}^{4} \sum_{n=1}^{m+1} Y_n^{\delta k+k-1} |S_i|^k$$

(By Minokowski's inequality)

Our Theorem will be established if we show that $\sum_{n=1}^{m+1} Y_n^{\delta k+k-11} |S_i|^k < \infty \ , \forall\, i = 1,2,3,4.$

$$\sum_{n=1}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} |S_1|^k = \sum_{n=1}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} \left| \sum_{r=1}^{n} A_{nr} \frac{P_r}{p_r} \Delta t_r \right|^k$$

$$\le \sum_{n=1}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} \sum_{r=1}^{n} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k A_{nr} \left(\sum_{r=1}^{n} A_{nr} \right)^{k-1}$$

(Using Holder's inequality)

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} A_{nr}, \text{ by (3.5)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} \left(f(\delta) \right)^k \left(\frac{P_n}{p_n} \right)^{k-1} A_{nr} \text{, by (3.3)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{1}{a_{rr}} \right)^k |\Delta t_r|^k a_{rr} \text{ , (using 3.4)}$$

$$= O(1) \sum_{r=1}^{m+1} X_r^{k-1} |\Delta t_r|^k \text{ , (using 3.2)}$$

$$= O(1).$$

Next

$$\sum_{n=1}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} |S_2|^k = \sum_{n=1}^{m+1} \left(f(\delta) \right)^k Y_n^{k-1} \left| \sum_{r=1}^{n} A_{nr} \frac{P_{r-2}}{p_{r-1}} \Delta t_{r-1} \right|^k$$

$$\leq \sum_{n=1}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1}\sum_{r=1}^{n}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k A_{nr}\left(\sum_{r=1}^{n} A_{nr}\right)^{k-1}$$

$$= O(1)\sum_{r=1}^{m+1}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k \sum_{n=r}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1} A_{nr}, \text{ by (3.5)}$$

$$= O(1)\sum_{r=1}^{m+1}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k \sum_{n=r}^{m+1}\left(f(\delta)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} A_{nr}, \text{ by (3.3)}$$

$$= O(1)\sum_{r=2}^{m+1}\left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k a_{rr}, \text{ (using 3.4)}$$

$$= O(1)\sum_{r=2}^{m+1}\left(\frac{1}{a_{rr}}\right)^k |\Delta t_r|^k a_{rr}$$

$$= O(1)\sum_{r=2}^{m+1} X_r^{k-1}|\Delta t_r|^k, \text{ (using 3.2)}$$

$$= O(1).$$

Also,

$$\sum_{n=1}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1}|S_3|^k = \sum_{n=1}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1}\left|\sum_{r=1}^{n-1} A_{n-1,r}\frac{P_r}{p_r}\Delta t_r\right|^k$$

$$\leq \sum_{n=1}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1}\sum_{r=1}^{n-1}\left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k A_{n-1,r}\left(\sum_{r=1}^{n} A_{n-1,r}\right)^{k-1}$$

$$= O(1)\sum_{r=1}^{m+1}\left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k \sum_{n=r+1}^{m+1}\left(f(\delta)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} A_{n-1,r}, \text{ using (3.5)}$$

$$= O(1)\sum_{r=1}^{m+1}\left(\frac{1}{a_{rr}}\right)^k |\Delta t_r|^k a_{rr}, \text{ (using 3.4)}$$

$$= O(1)\sum_{r=1}^{m+1} X_r^{k-1}|\Delta t_r|^k, \text{ (using 3.2)}$$

$$= O(1).$$

Finally,

$$\sum_{n=1}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1}|S_4|^k = \sum_{n=1}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1}\left|\sum_{r=1}^{n-1} A_{n-1,r}\frac{P_{r-2}}{p_{r-2}}\Delta t_{r-1}\right|^k$$

$$\leq \sum_{n=1}^{m+1}\left(f(\delta)\right)^k Y_n^{k-1}\sum_{r=1}^{n-1}\left(\frac{P_{r-2}}{p_{r-2}}\right)^k |\Delta t_{r-1}|^k A_{n-1,r}\left(\sum_{r=1}^{n} A_{n-1,r}\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{m+1}\left(f\left(\delta\right)\right)^k Y_n^{k-1}\sum_{r=1}^{n-1}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k \left|\Delta t_{r-1}\right|^k A_{n-1,r}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k \left|\Delta t_{r-1}\right|^k \sum_{n=r+1}^{m+1} A_{n-1,r}\left(f\left(\delta\right)\right)^k Y_n^{k-1}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{1}{a_{r-1,r-1}}\right)^k \left|\Delta t_{r-1}\right|^k a_{r-1,r-1}, \text{ (using 3.4)}$$

$$= O(1)\sum_{r=1}^{m} X_{r-1}^{k-1}\left|\Delta t_{r-1}\right|^k, \text{ (using 3.2)}$$

$$= O(1).$$

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